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Makeham's Generalised Distribution

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Abstract: The aim of this publication is to present a new probability distribution which for particular parameter values has a bimodal density function and a bathtub hazard rate function. All the main reliability properties of such distribution will be described in details in this paper. Such a distribution can constitute a very good mathematical model that would enable the description of the lifetime of technical devices. It can successfully be implemented in the planning of a burn-in procedure and a preventive maintenance of non-repairable devices.

Key words: Makeham's distribution, bathtub hazard rate function, bimodal density function

Symbols

MMD – Makeham's modified distribution
MGD – Makeham's generalised distribution
$h(t)$ – hazard rate function
$f(t)$ – density function
$F(t)$ – cumulative distribution function
$R(t)$ – reliability function

I. INTRODUCTION

Until 1980s the bathtub hazard rate function (with a flat minimum) was considered as an old failure pattern for the lifetime of technical devices. Several publications which appeared at the beginning of the 1980s [5-7, 15, 22, 29, 30] suggested that the exemplary distribution of the lifetime of technical devices, especially of electronic components, is a bimodal distribution. The first mode relates to failure that result from internal defects which are the consequence of material defects or imperfection of manufacturing process. The second mode, which is situated substantially further from the first one, relates to wear out processes. It was the so called new failure pattern. The old failure pattern, on the other hand, was connected with the bathtub hazard rate function. In this paper an attempt of MMD improvement is made with the ultimate goal of arriving a bimodal distribution. The bathtub hazard rate function is an necessary condition in a distribution of the lifetime of technical devices, whereas, the bimodal density function is a sufficient condition. A mathematical model based on the bathtub hazard rate function and a bimodal density function allows planning burn-in procedures and preventive maintenance of non-repairable devices. On the basis of a model with a decreasing hazard rate function it is possible to assess the optimal time of burn-in procedures for a given guarantee time, maximizing the probability of correct work during this period of time. For an increasing hazard rate function it is possible to determine the optimal time for preventive maintenance. By minimizing the sum of two factors: loss due to failures and costs of preventive maintenance the balancing of burn-in and preventive maintenance procedures based on MMD and MGD have been presented in detail in papers [19, 20], and those based on Weibull's distribution in paper [10].

The aim of this paper is to present a new reliability distribution having the bimodal density function and the bathtub hazard function. Compare it with the known distributions with bathtub hazard functions used in reliability theory so far. These are: generalized gamma distribution [26, 27], Firkowicz's distribution [12], beta distribution of the first kind [14], Muth's distribution [24], Urban's distribution [28], Weibull's generalized distribution [8], Kao distribution [16], Malik's distribution [23], Makeham's modified distribution (MMD) [4, 8, 19]. By implementing a shape function instead of a shape parameter into MMD (this idea is taken from Weibulls' generalized distribution) Makeham's generalized distribution (MGD) is obtained. This distribution becomes a bimodal distribution of a bathtub hazard function for certain parameter values.

The mentioned distributions: Firkowicz's distribution, beta distribution of the first kind or Muth's distribution are based on a limited time interval. They have the bathtub hazard function with a vertical asymptote. However, from a physics of failure it is difficult to explain as it includes the assumption that after a certain time all the devices will stop to work, whereas lifetime is assumed as a random variable. Moreover, Malik's and Kao distribution have a complicated mathematical form. They require calculation of large number of parameters. The generalized gamma distribution seems to be very flexible, yet, this flexibility is in this case rather a shortcoming than an advantage since the hazard function can be constant, bathtub, increasing or decreasing. In practice, the optimalization of burn-in procedures and preventive maintenance would include the values obtained from analyzing the data from failures (an estimation of these parameters) instead of parameter values that are actually known. However, the accuracy of parameter value estimation, and especially of shape parameters, by the generalized gamma distribution is relatively low [27]. Thus, it may be so that the hazard function of the observed devices can be of the bathtub type. The parameter estimate obtained from the sample are of such a value that the theoretical hazard function obtained by substituting the estimation to the final formula is not of a bathtub type. On the other hand, the gamma distribution is not flexible enough to obtain a bimodal density function from it. As far as Urban's distribution is concerned, the shortcoming is the fact that the scale parameter value influences on the shape of the hazard function. This distribution is only equipped with the scale parameter and a certain weight which significantly reduces the possibility of modeling of lifetime. What is more, it can only be expanded on the time axis. Finally, Weibull's generalised distribution which produces a bathtub hazard function can also give a bimodal density function though with certain restrictions regarding the range of its parameter values. This distribution is well-defined when a certain relation between the scale and shape parameters is fulfilled, otherwise the density function is negative in a certain range.

Thus, the MGD can be apply for planning burn-in and preventive maintenance procedures of non-repairable devices.

II. MAKEHAM'S MODIFIED DISTRIBUTION

The aim of this paper is to define the distribution of the bimodal density function and the bathtub hazard rate function. Thus, important characteristics of the distribution in question is the hazard rate function. It is a quotient of the density and the reliability function [2, 13, 17, 25]:

$$h(t) = \frac{f(t)}{R(t)}. \quad (1)$$

Hazard rate function is the conditional probability that failure of the device will occur during a small interval $<t; t + \Delta t>$ given that device has survived to time t.

The bathtub hazard rate function is typical for the lifetime description of most technical devices. At the beginning the failure rate decreases, after a while it becomes stable, then it increases due to natural wear out processes.

In literature [1, 3, 4, 11] Makeham's distribution is presented as an hazard rate function of this kind:

$$h(t) = \rho_1 \exp(\rho_2 t) + \rho_0 \quad (2)$$

where ρ_0 and ρ_1 are failure rate coefficients and ρ_2 is the scale parameter.

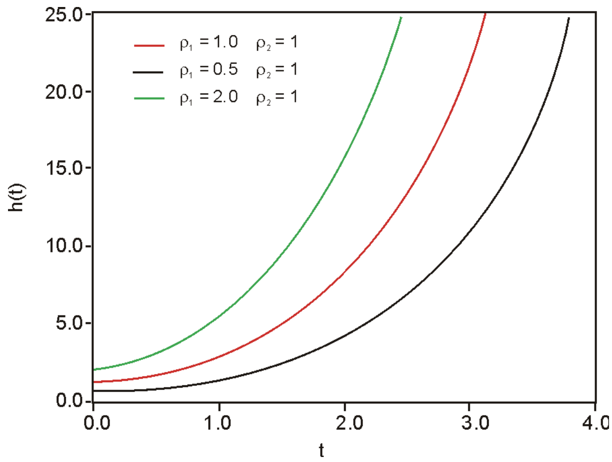

Fig. 1. The hazard rate function of Makeham's distribution for the particular parameter values

The hazard rate function of this distribution is strictly increasing. However, empirical distributions used for the description of the lifetime do usually have a bathtub hazard rate function, which is characterised by a flat minimum. By introducing (2) the shape parameter b to Makeham's distribution the hazard rate function will be of the following form:

$$h(t) = \rho_1 \cdot t^{b-1} \cdot \exp\left(\rho_2 t^b\right). \quad (3)$$

By applying a more popular form of the notation of the formula (3) the following is obtained:

$$h(t) = \frac{b}{a}\left(\frac{t}{a}\right)^{b-1} \cdot \exp\left[\left(\frac{t}{a}\right)^b\right], \quad (4)$$

where a is the scale parameter and b the shape parameter.

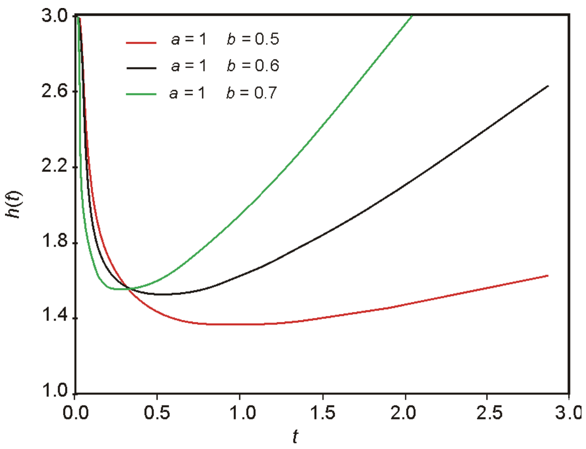

Fig. 2. The hazard rate function of Makeham's modified distribution for the particular parameter values

The distribution presented in this paper is called Makeham's modified distribution (MMD). The function (4) is the product of a monotonically decreasing and a monotonically increasing function. For $b < 1$ the hazard rate function has a bathtub curve. This is a very desirable characteristic of the reliability models. Such a distribution is described in [18, 20, 21]. MMD is here only the starting point to obtain a new reliability distribution with a bimodal density function.

III. MAIN CHARACTERISTICS OF THE MAKEHAM'S GENERALISED DISTRIBUTION

The cumulative distribution function of the MMD is of the following form:

$$F(t) = 1 - \exp\left[1 - \exp\left(\frac{t}{a}\right)^{b}\right]. \tag{5}$$

By introducing MMD the shaping function $b(t)$ we receive the bimodal density function with the bathtub hazard rate function. This idea was taken from [8].

Let the shaping function be a linear function of the following form:

$$b(t) = b_0 + b_1 t\,. \tag{6}$$

Then the cumulative distribution function will be expressed by means of this formula:

$$F(t) = 1 - \exp\left[1 - \exp\left(\frac{t}{a}\right)^{b_0 + b_1 \cdot t}\right], \tag{7}$$

where a is the scale parameter and b_0 and b_1 are the shape parameters. This distribution will be called Makeham's generalised distribution (MGD).

As it will show later a distribution so defined has not only a bathtub hazard rate function but also a bimodal density function (the condition is $b_0 \le 1$).

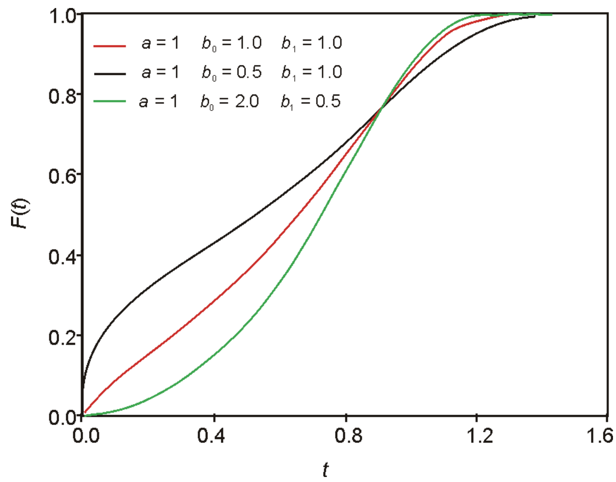

Fig. 3. The cumulative distribution function of MGD for the particular parameter values

IV. BASIC LIFETIME CHARACTERISTICS OF MGD

The cumulative distribution function is strictly connected with the reliability function of this form:

$$R(t) = 1 - F(t) \tag{8}$$

so for MGD it is:

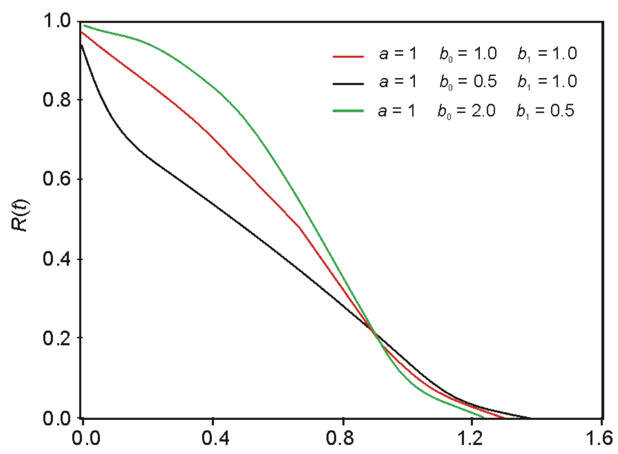

Fig. 4. The reliability function of MGD for the particular parameter values

$$R(t) = \exp\left[1 - \exp\left(\frac{t}{a}\right)^{b_0 + b_1 \cdot t}\right]. \tag{9}$$

Derivative of the cumulative distribution function (7) gives density function of the following form:

$$f(t) = \left(\frac{t}{a}\right)^{b_0 + b_1 \cdot t} \times$$

$$\times \left[b_1 \cdot \ln\left(\frac{t}{a}\right) + \frac{(b_0 + b_1 \cdot t)}{t}\right] \times \tag{10}$$

$$\times \exp\left[1 - \exp\left(\frac{t}{a}\right)^{b_0 + b_1 \cdot t} + \left(\frac{t}{a}\right)^{b_0 + b_1 \cdot t}\right].$$

The function (10) determines the probability density if:

$$b_1 \cdot \ln\left(\frac{t}{a}\right) + \frac{(b_0 + b_1 \cdot t)}{t} \geq 0. \tag{11}$$

By defining an auxiliary function:

$$u(t) = b_1 \cdot \ln\left(\frac{t}{a}\right) + b_1 + \frac{b_0}{t} \tag{12}$$

the derivative was established

$$u'(t) = \frac{b_1}{t} - \frac{b_0}{t^2}. \tag{13}$$

By solving the formula:

$$u'(t) = 0. \tag{14}$$

The result is that for:

$$t = \frac{b_0}{b_1} \tag{15}$$

this function can have an extreme. Thus if $u(t)$ is non-negative the following condition must be fulfilled:

$$u\left(\frac{b_0}{b_1}\right) = $$

$$= b_1 \cdot \ln\left(\frac{b_0}{b_1 \cdot a}\right) + 2b_1 \geq 0. \tag{16}$$

The condition is fulfilled if:

$$\frac{b_0}{b_1 a} \geq \frac{1}{e^2}. \tag{17}$$

Thus the function (10) determines the density only if the condition (17) is fulfilled.

The density function can be unimodal for $b_0 > 1$ that for particular parameter values is shown in Fig. 5

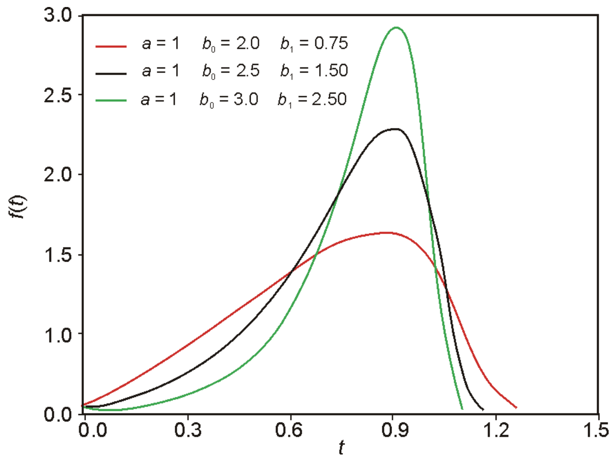

Fig. 5. The unimodal density function of MGD for the particular parameter values

However, for $b_0 \leq 1$ the density function of MGD can have two modes (the first mode at 0, the second mode substantially further) which is shown in Fig. 6.

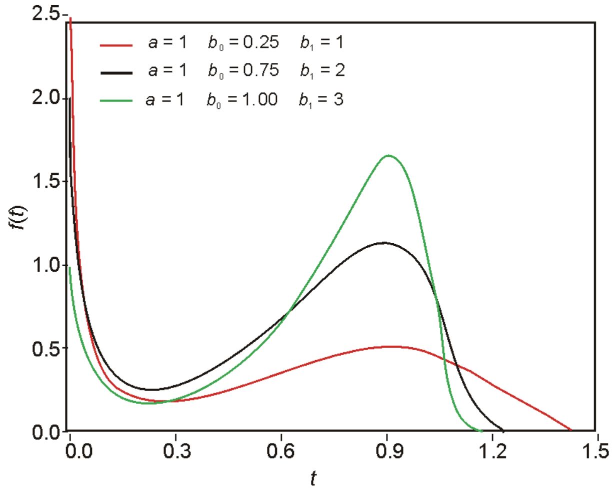

Fig. 6. The bimodal density function of MGD for the particular parameter values

We can determine the location of the mode only on numerical way. The results for the numerical treatment are presented in Fig. 7.

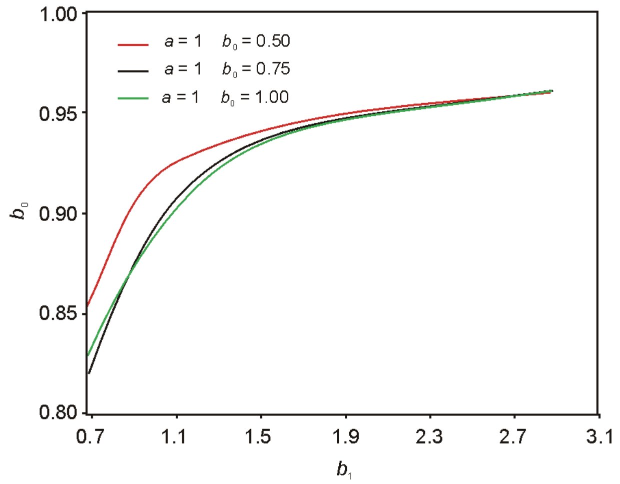

Fig. 7. Location of the second mode as a function of the parameter b_1 depending on the b_0 particular parameter values

As can be seen in the Fig. 7 the position of the mode is approaching one with the increase of the shape parameter.

By substituting formulas (10) and (9) into formula (1) the MGD hazard rate function is obtained:

$$h(t)=\left(\frac{t}{a}\right)^{b_0+b_1\cdot t}\left[b_1\ln\left(\frac{t}{a}\right)+\frac{(b_0+b_1t)}{t}\right]\times \cdot \times\exp\left(\frac{t}{a}\right)^{b_0+b_1\cdot t}. \quad (18)$$

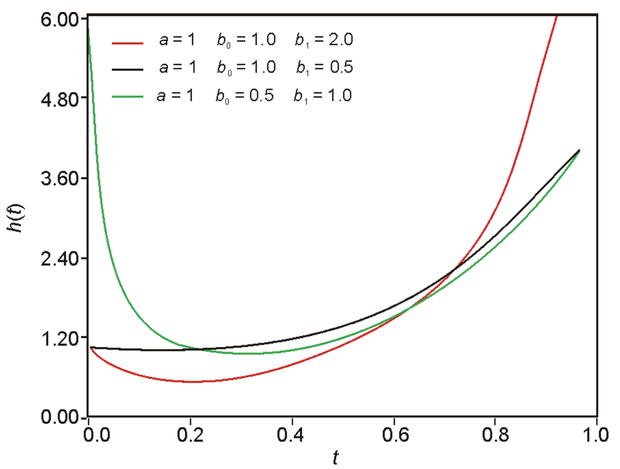

Fig. 8. MGD hazard rate function for the particular parameter values

The MGD hazard rate function has a minimum only for $b_0 \le 1$. They can be determined numerically. The results for the numerical treatment are presented in Fig. 9.

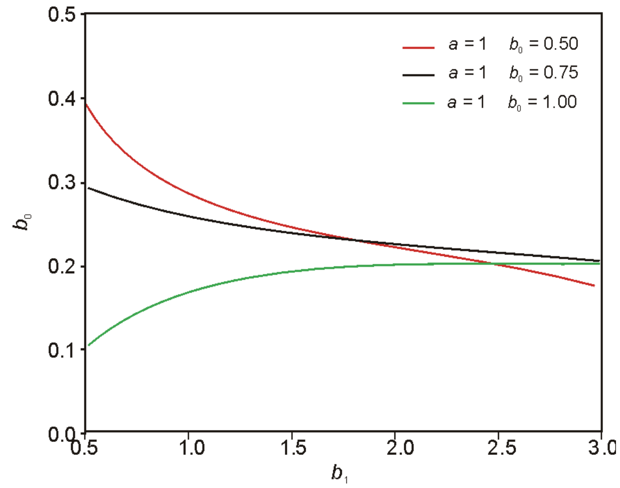

Fig. 9. Minimum of the MGD hazard rate function as a function of the parameter b_1 depending on the b_0 particular parameter values

The minimum of the MGD hazard rate function as a function of the parameter b_1 are decreasing for $a = 1$, $b_0 = 0.5$ and $b_0 = 0.75$ and slowly increasing for $a = 1$ and $b_0 = 1$.

In practice it is easier to determine a hazard rate average function [1], of the following form:

$$h_{\acute{s}}(t)=\frac{1}{t}\int_0^t h(u)du\,. \quad (19)$$

Thus substituting (18) into (19) the formula for the hazard rate average function for MGD is obtained:

$$h_s(t)=-\frac{1}{t}\cdot\left[1-\exp\left(\frac{t}{a}\right)^{b_0+b_1t}\right]. \quad (20)$$

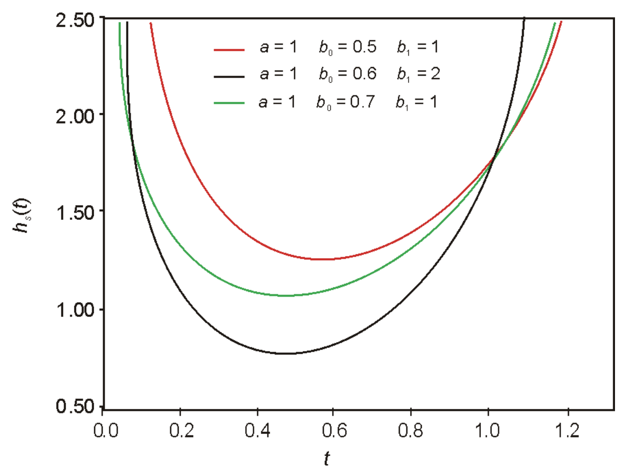

Fig. 10. The MGD hazard rate averaging function for particular parameter values

V. THE MOMENTS AND RELATED CHARACTERISTICS OF MGD

When examining numerical characteristics of MGD, the formula for MGD's ordinary moments was taken into consideration. They are determined by the following formula:

$$m_k = \int_0^\infty t^k \left(\frac{t}{a}\right)^{b_0+b_1 t} \left[b_1 \ln\frac{t}{a} + \frac{b_0 + b_1 t}{t}\right] \times \tag{21}$$

$$\times \exp\left[1 - \exp\left(\frac{t}{a}\right)^{b_0+b_1 t} + \left(\frac{t}{a}\right)^{b_0+b_1 t}\right] dt\,.$$

In this paper the first ordinary moment, which most common is called the expected value, defines the mean time to failure. The mean time to failure of the MGD distribution depending on the parameters b_0, b_1 is shown graphically in Fig. 11, Fig. 12

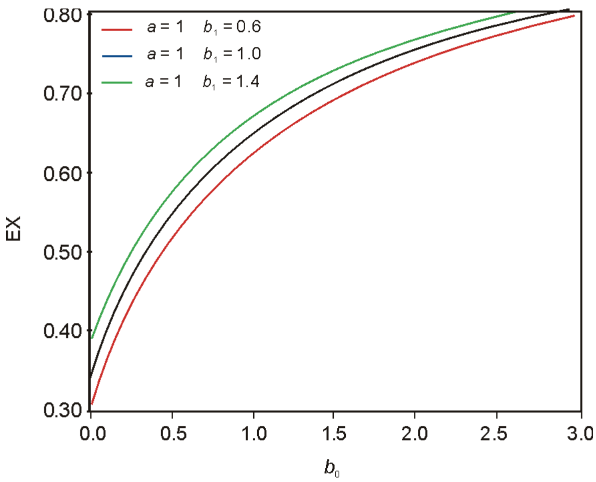

Fig. 11. The mean time to failure as a function of the parameter b_0

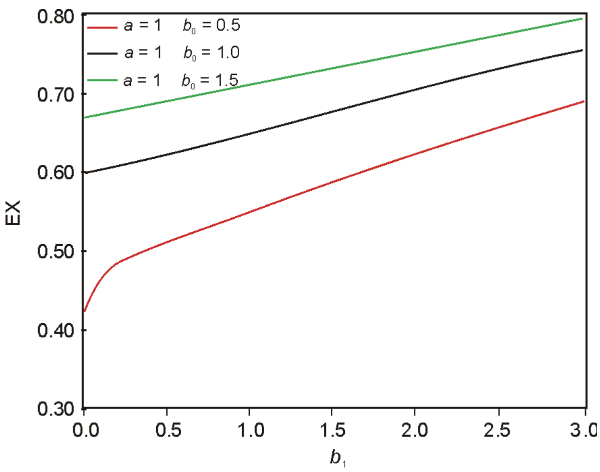

Fig. 12. The mean time to failure as a function of the parameter b_1

Figure 11 shows that the mean time to failure as a function of the parameter b_0 increases when parameter values of b_0 increases.

Figure 12 shows that the mean time to failure as a function of the parameter b_1 increases when parameter values of b_1 increases.

Most common numerical characteristics of distribution is also standard deviation. It is shown graphically for MGD in Fig. 13 and Fig. 14.

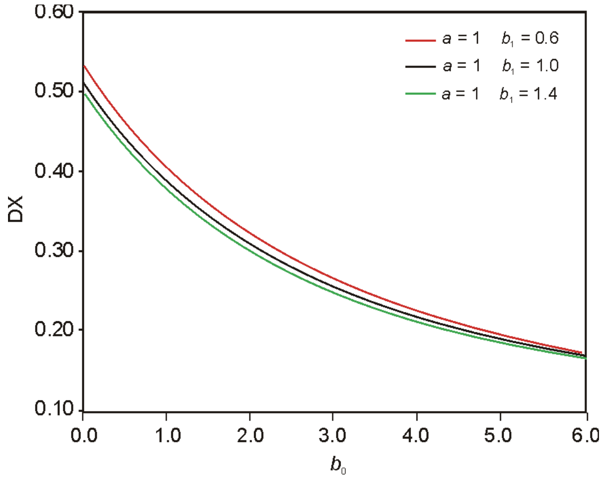

Fig. 13. Standard deviation as a function of the parameter b_0

Standard deviation as a function of the parameter b_0 decreases when parameter values of b_0 increases.

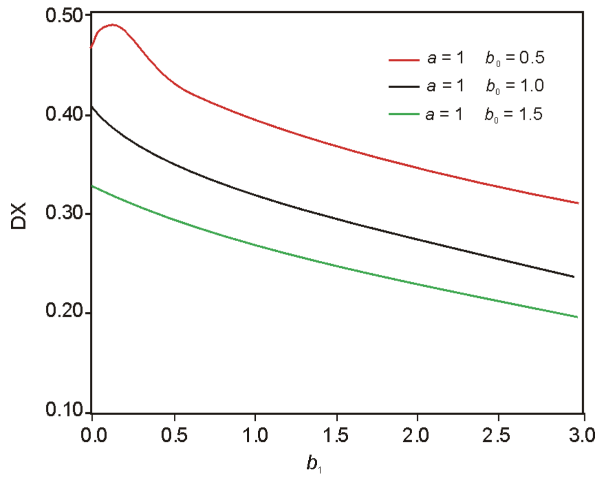

Fig. 14. Standard deviation as a function of the parameter b_1

Standard deviation as a function of the parameter b_1 decreases when parameter values of b_1 increases.

The estimation of the MGD parameters by means of classical analytical methods may result impossible due to the intrication of the distribution form and as many as three parameters for estimation.

VI. NUMERICAL EXAMPLE

Example

Tabele 1 shows the lifetime of a certain device. Check whether the lifetime of this device followed the MGD distribution with the estimated parameter values.

Table 1. The lifetime of a certain device

i	t_i	i	t_i
1	0.0094	16	10.0192
2	0.0500	17	10.4077
3	0.4064	18	10.4791
4	4.6307	19	11.0706
5	5.1741	20	11.3250
6	5.8808	21	11.5284
7	6.3348	22	11.9226
8	7.1654	23	12.0294
9	7.2316	24	12.0740
10	8.2604	25	12.1835
11	9.2962	26	12.3549
12	9.3812	27	12.5381
13	9.5223	28	12.8049
14	9.8783	29	13.4615
15	9.9346	30	13.8530

Source – Data obtained by the author

Firstly, the MGD parameters were estimated. In case of so many parameters even the maximum likelihood method does not provide satisfactory values of the estimations [27].

Thus, in order to estimate the parameters of this distribution the least squares method was used. The method consists in the minimalisation of the squared deviations of the experimental cumulative distribution function from the theoretical cumulative distribution function by means of Excel Solver. After performing all the necessar calculations the estimated parameter values are a = 12.3586, b_0 = 0.5472, b_1 = 0.2736. The null hypothesis states that the data from Tab. 1 can be described by means of MGD with parameters a = 12.3586, b_0 = 0.5472, b_1 = 0.2736. This hypothesis was verified using the Mizes-Smirnov test. A computer implementation of this test can be found in [9]. The empirical value of the test statistics is $n\omega^2 = 0.01863$. In comparison with the Mizes-Smirnow critical values it appears that at the level of significance $\alpha = 0.01$ there exists no foundation for discarding the null hypothesis. Thus the data from Tab. 1 can be described by means of MGD with the estimated parameter values.

The hazard rate function and the MGD density function for the estimated parameter values of the distribution.

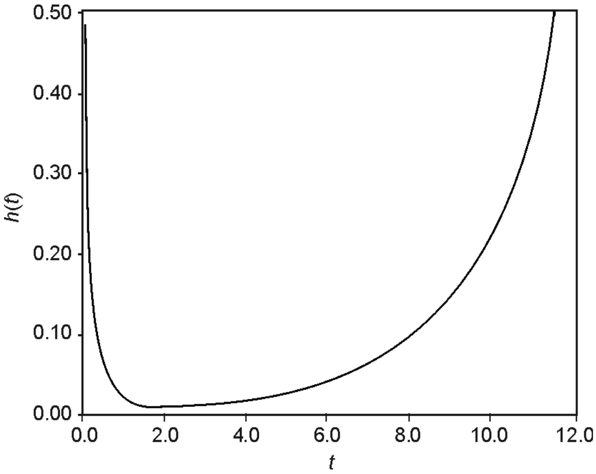

Fig. 15. The MGD hazard rate function with the estimated parameter values

The hazard rate function fulfils the necessary condition of the distribution for the description of the lifetime of technical devices because it has the desired bathtub characteristic. It reaches minimum with t = 1.962.

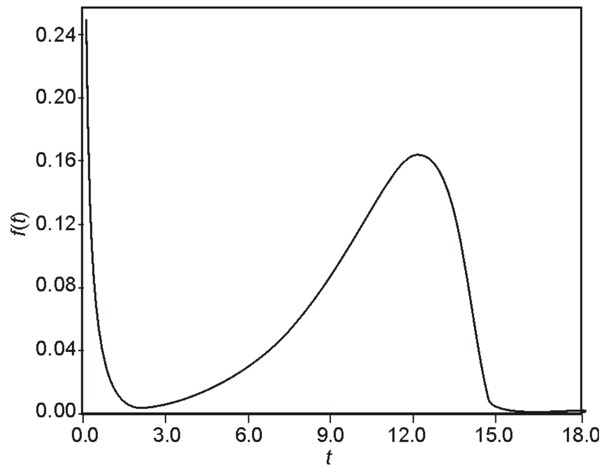

Fig. 16. The MGD density function for the estimated parameter values

The density function fulfils the sufficient condition of the distribution that describes the lifetime of the technical devices because it is bimodal (the first mode at zero). The second reaches maximum at t = 11.907.

On the basis of Fig. 15 and Fig. 16 it can be concluded that the analysed device most probably can be subject to a breakdown at the beginning of its use. It could be the result of a latent defect acquired during the production process. If the damage does not occur until the time t = 1.962 then until time t = 11.907 its failure probability will be decreasing. It is a typical using time. After time t = 11.907 a reduction in the efficiency of the device may be noticeable, thus the probability of failure will be increasing. It is a result of wear out processes of the device.

VII. CONCLUSION

MMD is a distribution whose hazard function is bathtub for certain parameter values. This is highly desirable in the modeling of the lifetime of technical devices. Yet, it is such a flexible distribution that by implementing a linear shape function, instead of a shape parameter, MGD is obtained. The MGD hazard function is bathtub for particular parameter values and the density function is bimodal which means that the distribution fulfills both the necessary condition of a distribution for modeling of the lifetime of technical devices and the sufficient condition as was proven in the 1980s. The considerable advantage of the above presented distribution is its simple analytical form and a small number of parameters for calculation which is crucial in small random sample commonly used in reliability research.

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